

# SPECTRAL THEOREM FOR QUATERNIONIC NORMAL OPERATORS : MULTIPLICATION FORM

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**ABSTRACT.** In this article we prove the multiplication form of a spectral theorem for normal operators in quaternionic Hilbert spaces: Let  $T$  be a right linear normal operator in a quaternionic Hilbert space  $H$  with the domain  $\mathcal{D}(T) \subset H$ . We prove that for a fixed unit imaginary quaternion, say  $m$ , there exists a Hilbert basis  $\mathcal{N}_m$  of  $H$ , a measure space  $(\Omega, \mu)$ , a unitary operator  $U: H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  and a  $\mu$ -measurable function  $\phi: \Omega \rightarrow \mathbb{C}_m$ , ( here  $\mathbb{C}_m = \{\alpha + m\beta; \alpha, \beta \in \mathbb{R}\}$ ) so that if  $T$  is expressed with respect to  $\mathcal{N}_m$  then

$$Tx = U^* M_\phi Ux, \text{ for all } x \in \mathcal{D}(T),$$

where  $M_\phi$  is the multiplication operator in  $L^2(\Omega; \mathbb{H}; \mu)$  induced by  $\phi$  with  $U(\mathcal{D}(T)) \subseteq \mathcal{D}(M_\phi)$ .

In the process, we prove that every complex Hilbert space is a slice Hilbert space of some quaternionic Hilbert space. Also, we prove that there exists an anti self-adjoint, unitary operator which commutes with an unbounded normal operator in a quaternionic Hilbert space. In proving all these results, we solve the problem by reducing the problem to the complex case and lift it to the quaternionic case.

## 1. INTRODUCTION

The theory of quaternionic Hilbert spaces is found to be useful in quaternionic quantum mechanics. Particularly, in mathematical physics, finding the solution of the Schrodinger equation over the skew-field of quaternions is equivalent to the study of diagonalization of an anti self-adjoint, unitary operator on a quaternionic Hilbert space [6].

In quantum mechanics most of the operators which we come across are unbounded. In particular, this is the case in quaternionic quantum mechanics also.

One of the most important operators in quantum mechanic is the position operator, which is nothing but a multiplication operator defined in a Hilbert space. This is a normal operator. In fact, it is well known in the classical theory of operators that every normal operator is a multiplication operator induced by a suitable function. One can ask whether the same is true or not in quaternionic setting.

In this direction, upto our knowledge, only [12] dealt with such a question and the solution of this is given by using spectral system. In this article, the author used the symplectic image concept to obtain the result.

This question is answered for normal operators in a real Hilbert spaces by S. H. Kulkarni and S. Agarwal in [1] using Banach algebra techniques. Both the above

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results do not use the concept of the spherical spectrum of a quaternionic normal operator.

In our case we show that an unbounded normal operator on a quaternionic Hilbert space is unitarily equivalent to a multiplication operator induced by an appropriate function. The result exactly looks like the classical one. To our surprise, we observed that the function which determine the multiplication operator is  $\mathbb{C}_m$ -valued, for a fixed unit imaginary quaternion  $m$ .

In proving our results we use the following approach: Given an unbounded normal operator in a quaternionic Hilbert space, we associate a unique operator in a complex Hilbert space, which have similar properties and by using the results of the complex operator theory, we obtain the results for quaternionic operator. Though this technique is available in the literature (see [7] for details), to our knowledge we use this for the first time. Later, we extend this technique for quaternionic operators defined between two different Hilbert spaces.

We organize this article in four sections. In the second section we recall basic properties of the ring of quaternions, quaternionic Hilbert spaces and operators in quaternionic Hilbert spaces. In the third section, we prove two main results:

- we show that every complex Hilbert space is a slice Hilbert space of a quaternionic Hilbert space for some anti self-adjoint, unitary operator and
- a linear operator between two complex Hilbert spaces can be extended to a unique right linear operator between quaternionic Hilbert spaces.

In the fourth section, we prove the spectral theorem (multiplication form) for right linear bounded operators on quaternionic Hilbert spaces. In the final section, we prove the similar results for unbounded normal operator in quaternionic Hilbert space by using the  $\mathcal{Z}$ -transform. In this proof we restrict the operator to the slice Hilbert space, then associate it to the  $\mathcal{Z}$ -transform to get the spectral theorem for the operator on the slice Hilbert space. Finally, we lift this result to the quaternionic case.

## 2. PRELIMINARIES

Let  $\mathbb{H}$  denotes the ring of all real quaternions. If  $q \in \mathbb{H}$ , then  $q = q_0 + q_1i + q_2j + q_3k$ , where  $q_\ell \in \mathbb{R}$  for each  $\ell = 0, 1, 2, 3$ . Here  $i, j, k$  satisfy

$$i^2 = j^2 = k^2 = -1 \text{ and } i \cdot j \cdot k = -1.$$

For  $q \in \mathbb{H}$ , the conjugate of  $q$  is  $\bar{q} = q_0 - q_1i - q_2j - q_3k$ . The real part of  $\mathbb{H}$  is denoted by  $\text{Re}(\mathbb{H}) = \{q \in \mathbb{H} : q = \bar{q}\}$  and the imaginary part of  $\mathbb{H}$  is denoted by  $\text{Im}(\mathbb{H}) = \{q \in \mathbb{H} : q = -\bar{q}\}$ . The set  $\mathbb{S} := \{q \in \text{Im}(\mathbb{H}) : |q| = 1\}$  is the imaginary unit sphere. For  $p, q \in \mathbb{H}$ , we have  $\overline{p \cdot q} = \bar{q} \cdot \bar{p}$  and  $|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ .

For  $m \in \mathbb{S}$ , the field  $\mathbb{C}_m := \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$  is called a slice of  $\mathbb{H}$ . Here  $\mathbb{C}_m$  is isomorphic to the complex field  $\mathbb{C}$  through the mapping  $\alpha + m\beta \rightarrow \alpha + i\beta$ . For  $m \neq \pm n \in \mathbb{S}$ , we have  $\mathbb{C}_m \cap \mathbb{C}_n = \mathbb{R}$ . We also have  $\mathbb{H} = \bigcup_{m \in \mathbb{S}} \mathbb{C}_m$ . The positive upper half plane of  $\mathbb{C}_m$  is defined by  $\mathbb{C}_m^+ = \{\alpha + m\beta; \alpha \in \mathbb{R}, \beta \geq 0\}$ .

**Note 2.1.** All the results in complex Hilbert spaces holds true in the slice  $\mathbb{C}_m$ -Hilbert spaces, for any  $m \in \mathbb{S}$ .

Relation on  $\mathbb{H}$ : Define a relation on  $\mathbb{H}$  by

$$p \sim q \Leftrightarrow p = s^{-1}qs, \text{ for some } 0 \neq s \in \mathbb{H}.$$

Here  $\sim$  is an equivalence relation on  $\mathbb{H}$ . The equivalence class of  $q$  is given by

$$[q] := \{s^{-1}qs : 0 \neq s \in \mathbb{H}\}.$$

If  $\text{Im}(q) \neq 0$ , then  $q = \text{Re}(q) + \frac{\text{Im}(q)}{|\text{Im}(q)|} \cdot |\text{Im}(q)|$ . It follows that

$$[q] = \text{Re}(q) + |\text{Im}(q)| \cdot \mathbb{S}.$$

We have the following observations through the representation of the equivalence classes:

$$(1) \quad \text{for all } m \in \mathbb{S}, [q] \cap \mathbb{C}_m = \left\{ \text{Re}(q) \pm m \cdot \frac{\text{Im}(q)}{|\text{Im}(q)|} \right\},$$

$$(2) \quad p \sim q \Leftrightarrow \text{Re}(p) = \text{Re}(q) \text{ and } |\text{Im}(p)| = |\text{Im}(q)|.$$

**Definition 2.1.** [7, Definition 2.3] *Let  $H$  be a right  $\mathbb{H}$ -module. A map*

$$\langle \cdot | \cdot \rangle : H \times H \longrightarrow \mathbb{H}$$

*satisfying:*

- (1) *If  $u \in H$ , then  $\langle u | u \rangle = 0 \Leftrightarrow u = 0$*
- (2)  *$\langle u | v + w \cdot q \rangle = \langle u | v \rangle + \langle u | w \rangle \cdot q$ , for all  $u, v \in H$  and  $q \in \mathbb{H}$*
- (3)  *$\langle u | v \rangle = \overline{\langle v | u \rangle}$ , for all  $u, v \in H$ ,*

*is called an inner product on  $H$ . If we define  $\|u\|^2 = \langle u | u \rangle$ , for all  $u \in H$ , then  $\|\cdot\|$  is a norm on  $H$  and is called the norm induced by  $\langle \cdot | \cdot \rangle$ . If  $(H, \|\cdot\|)$  is complete space then it is called a right quaternionic Hilbert space.*

**Note 2.2.** *Throughout the article  $H$  denotes a right quaternionic Hilbert space.*

**Definition 2.2.** *Let  $(\Omega, \mu)$  be a measure space and fix  $m \in \mathbb{S}$ . If  $\phi : \Omega \rightarrow \mathbb{C}_m$  is measurable, then*

- (1) *essential supremum of  $\phi$  is defined by*

$$\text{ess sup}(\phi) = \inf \{ \alpha : \mu(\{x : \phi(x) > \alpha\}) = 0 \}$$

- (2) *essential range of  $\phi$  is defined by*

$$\text{ess ran}(\phi) := \{ \lambda : \forall \epsilon > 0, \mu(\{x : |\phi(x) - \lambda| = 0\}) > \epsilon \}$$

- (3)  *$\phi$  is said to be essentially bounded if  $\text{ess sup}(|\phi|)$  is finite.*

**Proposition 2.3.** [7, Proposition 2.5] *Let  $\mathcal{N}$  be a subset of a right quaternionic Hilbert space  $H$  such that, for  $z, z' \in \mathcal{N}$ ,  $\langle z | z' \rangle = 0$  if  $z \neq z'$  and  $\langle z | z \rangle = 1$ . The following conditions are equivalent:*

- (1) *For every  $x, y \in H$ , it holds:*

$$\langle x | y \rangle = \sum_{z \in \mathcal{N}} \langle x | z \rangle \langle z | y \rangle.$$

*The above series converges absolutely*

- (2) *For every  $x \in H$ , it holds:*

$$\|x\|^2 = \sum_{z \in \mathcal{N}} |\langle z | x \rangle|^2$$

- (3)  $\mathcal{N}^\perp = \{0\}$

- (4)  $\text{span } \mathcal{N} = H$ .

**Proposition 2.4.** [7, Proposition 2.6] *Every quaternionic Hilbert space  $H$  admits a subset  $\mathcal{N}$ , called Hilbert basis of  $H$  such that, for  $z, z' \in \mathcal{N}$ ,  $\langle z|z' \rangle = 0$  if  $z \neq z'$  and  $\langle z|z \rangle = 1$ , and  $\mathcal{N}$  satisfies equivalent conditions stated in proposition 2.3.*

*Furthermore, if  $\mathcal{N}$  is Hilbert basis of  $H$  then every  $x \in H$  can be uniquely decomposed as follows:*

$$x = \sum_{z \in \mathcal{N}} z \langle z|x \rangle.$$

**Definition 2.5.** *Let  $(\Omega, \mu)$  be a measure space. Then*

$$L^2(\Omega; \mathbb{H}; \mu) := \left\{ f: \Omega \rightarrow \mathbb{H} \mid \int_{\Omega} |f(x)|^2 d\mu(x) < \infty \right\}$$

*is a right quaternionic Hilbert space with the inner product*

$$\langle f|g \rangle = \int_{\Omega} \overline{f(x)} \cdot g(x) d\mu(x).$$

*For a fixed  $m \in \mathbb{S}$ ,*

$$L^2(\Omega; \mathbb{C}_m; \mu) = \left\{ f: \Omega \rightarrow \mathbb{C}_m \mid \int_{\Omega} |f(x)|^2 d\mu(x) < \infty \right\}$$

*is a  $\mathbb{C}_m$ - Hilbert space with the inner product*

$$\langle f|g \rangle = \int_{\Omega} \overline{f(x)} \cdot g(x) d\mu(x).$$

**Definition 2.6.** [7, Definition 2.9] *Let  $(H_1, \|\cdot\|_1)$  and  $(H_2, \|\cdot\|_2)$  be two Hilbert spaces. A map  $T: H_1 \rightarrow H_2$  is said to be a right  $\mathbb{H}$ - linear operator if*

$$T(u \cdot q + v) = T(u) \cdot q + T(v),$$

*for every  $u, v \in H_1$  and  $q \in \mathbb{H}$ . We say that  $T$  is bounded or continuous, if there exists  $K > 0$  such that*

$$\|Tu\|_2 \leq K\|u\|_1, \text{ for all } u \in H_1.$$

*If  $T$  is bounded, then*

$$\|T\| = \sup \{\|Tu\|_2 : u \in H_1, \|u\|_1 = 1\}$$

*is finite and it is called the norm of  $T$ .*

We denote the set of all bounded right  $\mathbb{H}$ - linear operators between  $H_1$  and  $H_2$  by  $\mathcal{B}(H_1, H_2)$  and  $\mathcal{B}(H, H) = \mathcal{B}(H)$ . If  $T \in \mathcal{B}(H_1, H_2)$ , the null space and the range space are denoted by  $N(T)$  and  $R(T)$  respectively.

**Definition 2.7.** [7, Definition 2.12] *Let  $T \in \mathcal{B}(H)$ . Then there exists a unique operator  $T^* \in \mathcal{B}(H)$  such that  $\langle u|Tv \rangle = \langle T^*u|v \rangle$ , for all  $u, v \in H$ . This operator  $T^*$  is called the adjoint of  $T$ .*

**Definition 2.8.** [7, Definition 2.12] *Let  $T \in \mathcal{B}(H)$ . Then  $T$  is said to be*

- (1) *self-adjoint if  $T = T^*$*
- (2) *positive if  $T = T^*$  and  $\langle x|Tx \rangle \geq 0$ , for all  $x \in H$*
- (3) *anti self-adjoint if  $T^* = -T$*
- (4) *normal if  $TT^* = T^*T$*

(5) unitary if  $TT^* = T^*T = I$ .

**Example 2.9.** [12, Example 1.1] Let  $(\Omega, \mu)$  be  $\sigma$ -additive measure space and  $\phi: \Omega \rightarrow \mathbb{C}_m$  be  $\mu$ -measurable. Then we define the multiplication operator  $M_\phi: L^2(\Omega; \mathbb{H}; \mu) \rightarrow L^2(\Omega; \mathbb{H}; \mu)$ , by

$$M_\phi(g)(x) = \phi(x) \cdot g(x), \text{ for all } g \in L^2(\Omega; \mathbb{H}; \mu).$$

Then

- (1)  $M_\phi$  is a right  $\mathbb{H}$ -linear normal.
- (2)  $M_\phi^* = M_{\bar{\phi}}$
- (3)  $M_\phi \in \mathcal{B}(L^2(\Omega, \mathbb{H}, \mu))$  if and only if  $\phi$  is essentially bounded
- (4)  $\|M_\phi\| = \text{ess sup}(|\phi|)$
- (5)  $M_\phi^* = M_\phi \Leftrightarrow \phi = \bar{\phi}$  in  $\mu$ -a.e.

In the forthcoming sections we show that every normal operator in quaternionic Hilbert space is some form of multiplication operator.

**Definition 2.10.** Let  $T \in \mathcal{B}(H)$ . A closed subspace  $M$  of  $H$  is said to be invariant under  $T$  if  $T(M) := \{Tu: u \in M\} \subseteq M$ . Moreover, if  $M^\perp$  is also invariant under  $T$  then we say that  $M$  is a reducing subspace for  $T$ .

We recall the notion of the spherical spectrum of a right quaternionic linear operator in quaternionic Hilbert space.

**2.3. Spherical spectrum:** [7, Definition 4.1] Let  $T: H \rightarrow H$  be a right linear operator with domain  $\mathcal{D}(T)$ , a right linear subspace of  $H$  and  $q \in \mathbb{H}$ . Define  $\Delta_q(T): \mathcal{D}(T^2) \rightarrow H$  by

$$\Delta_q(T) := T^2 - T(q + \bar{q}) + I \cdot |q|^2.$$

The spherical resolvent of  $T$  is denoted by  $\rho_S(T)$  and is the set of all  $q \in \mathbb{H}$  satisfying the following three properties:

- (1)  $N(\Delta_q(T)) = \{0\}$
- (2)  $R(\Delta_q(T))$  is dense in  $H$
- (3)  $\Delta_q(T)^{-1}: R(\Delta_q(T)) \rightarrow \mathcal{D}(T^2)$  is bounded.

Then the spherical spectrum of  $T$  is defined by setting  $\sigma_S(T) = \mathbb{H} \setminus \rho_S(T)$ .

**Theorem 2.11.** [7, Theorem 5.9] Let  $T \in \mathcal{B}(H)$  be normal. Then there exists three mutually commuting bounded operators  $A, B$  and  $J$  on  $H$  such that

$$T = A + J.B,$$

where  $A = \frac{T+T^*}{2}$ ,  $B = \frac{|T-T^*|}{2}$  and  $J$  is an anti self-adjoint, unitary operator.

**Note 2.4.** Throughout this article,  $J$  denotes an anti self-adjoint, unitary operator.

**Lemma 2.12.** [8, Lemma 4.1] Let  $\langle \cdot | \cdot \rangle: H \times H \rightarrow \mathbb{H}$  be inner product on  $H$  and fix  $m \in \mathbb{S}$ . Define

$$H_\pm^{Jm} = \{u \in H : J(u) = \pm u \cdot m\}.$$

Then

- (1)  $H_\pm^{Jm} \neq \{0\}$  and the restriction of the inner product  $\langle \cdot | \cdot \rangle$  to  $H_\pm^{Jm}$  is  $\mathbb{C}_m$ -valued. Therefore  $H_\pm^{Jm}$  is  $\mathbb{C}_m$ -Hilbert space, called the slice Hilbert space of  $H$
- (2)  $H = H_+^{Jm} \oplus H_-^{Jm}$ .

**Remark 2.13.** If  $\mathcal{N}$  is Hilbert basis of  $H_+^{Jm}$ , then  $\mathcal{N}$  is also a Hilbert basis for  $H$  and it holds:

$$J(x) = \sum_{z \in \mathcal{N}} z \cdot m \langle z|x \rangle.$$

(see [7, Proposition 3.8(f)] for details).

For every  $m \in \mathbb{S}$ ,  $H_+^{Jm}$  is  $\mathbb{C}_m$ - Hilbert space. Our intension is to prove the converse statement: If  $K$  is  $\mathbb{C}_m$  - Hilbert space for some  $m \in \mathbb{S}$ , then there exists right quaternionic Hilbert space  $H$  and an anti self-adjoint, unitary  $J \in \mathcal{B}(H)$  such that  $K = H_+^{Jm}$ .

### 3. EXTENSION OF $\mathbb{C}_m$ -HILBERT SPACE TO A QUATERNIONIC HILBERT SPACE

It is known that a  $\mathbb{C}_m$ - linear operator in a slice Hilbert space  $H_+^{Jm}$  can be extended uniquely to a right linear operator in  $H$  [7]. In this section we prove that a linear operator in any  $\mathbb{C}_m$ - Hilbert space  $K$  can be extended uniquely to a quaternionic linear operator in some quaternionic Hilbert space  $H$  associated to  $K$ .

This is possible by showing that every  $\mathbb{C}_m$ - Hilbert space is a slice Hilbert space of some quaternionic Hilbert space, with respect to some anti self-adjoint, unity operator  $J$ .

**Lemma 3.1.** *Let  $H$  be right quaternionic Hilbert space. Then  $H$  is separable if and only if  $H_+^{Jm}$  is separable, for any  $m \in \mathbb{S}$  and any anti self-adjoint and unitary operator  $J$  on  $H$ .*

*Proof.* For any  $J$  and  $m \in \mathbb{S}$ , the space  $H_+^{Jm}$  is a closed subset of  $H$ . If  $H$  is separable, then  $H_+^{Jm}$  is also separable. It is clear from the definition of  $H_\pm^{Jm}$  that for any  $x \in H$ , we have

$$x \in H_+^{Jm} \Leftrightarrow x \cdot n \in H_-^{Jm}.$$

If  $H_+^{Jm}$  has countable dense subset say  $D_+$ , then  $D_- := \{x \cdot n : x \in D_+\}$  is countable and dense subset of  $H_-^{Jm}$ . By [7, Lemma 3.10],  $D_+ \oplus D_-$  is countable dense subset of  $H$ . This implies that  $H$  is separable.  $\square$

**Lemma 3.2.** *If  $x \in H_+^{Jm}$  and  $y \in H_-^{Jm}$ , then  $\langle x|y \rangle + \langle y|x \rangle = 0$ .*

*Proof.* It is clear from the proof of [7, Proposition 3.11(a)].  $\square$

We recall that every  $\mathbb{C}_m$ - linear operator in  $H_+^{Jm}$  can be extended uniquely to a right linear operator in  $H$ . Also the converse is possible with some assumptions, which are given below.

**Proposition 3.3.** [7, Proposition 3.11] *If  $T: \mathcal{D}(T) \subset H_+^{Jm} \rightarrow H_+^{Jm}$  is a  $\mathbb{C}_m$ - linear operator, then there exists a unique right  $\mathbb{H}$ - linear operator  $\tilde{T}: \mathcal{D}(\tilde{T}) \subset H \rightarrow H$  such that  $\mathcal{D}(\tilde{T}) \cap H_+^{Jm} = \mathcal{D}(T)$ ,  $J(\mathcal{D}(\tilde{T})) \subset \mathcal{D}(\tilde{T})$  and  $\tilde{T}(u) = T(u)$ , for every  $u \in H_+^{Jm}$ . The following facts holds:*

- (1) *If  $T \in \mathcal{B}(H_+^{Jm})$ , then  $\tilde{T} \in \mathcal{B}(H)$  and  $\|\tilde{T}\| = \|T\|$*
- (2)  *$J\tilde{T} = \tilde{T}J$ .*

*On the other hand, let  $V: \mathcal{D}(V) \rightarrow H$  be a right linear operator. Then  $V = \tilde{U}$ , for a unique bounded  $\mathbb{C}_m$ - linear operator  $U: \mathcal{D}(V) \cap H_+^{Jm} \rightarrow H_+^{Jm}$  if and only if  $J(\mathcal{D}(V)) \subset \mathcal{D}(V)$  and  $JV = VJ$ .*

*Furthermore,*

- (1) If  $\overline{\mathcal{D}(T)} = H_+^{Jm}$ , then  $\overline{\mathcal{D}(\tilde{T})} = H$  and  $(\tilde{T})^* = \tilde{T}^*$
- (2) If  $S: \mathcal{D}(S) \subset H_+^{Jm} \rightarrow H_+^{Jm}$  is  $\mathbb{C}_m$ -linear, then  $\widetilde{ST} = \tilde{S}\tilde{T}$
- (3) If  $S$  is the inverse of  $T$ , then  $\tilde{S}$  is the inverse of  $\tilde{T}$ .

**Remark 3.4.** In particular, if  $T \in \mathcal{B}(H)$  is non self-adjoint, normal, there exist an anti self-adjoint, unitary  $J \in \mathcal{B}(H)$  such that  $TJ = JT$  (see [7, Theorem 5.9]). If  $T \in \mathcal{B}(H)$  is self-adjoint, then the existence of such  $J$  is given in [7, Theorem 5.7(b)]. Hence all the results of Proposition 3.3 holds true for normal operators.

**Remark 3.5.** Let  $q \in \mathbb{H}$  and  $T: H_+^{Jm} \rightarrow H_+^{Jm}$  be  $\mathbb{C}_m$ -linear. Then by Proposition 3.3(2), we have

$$(3) \quad \Delta_q(\tilde{T}) = \tilde{\Delta}_q(T),$$

where  $\tilde{\Delta}_q(T)$  denotes the extension of  $\Delta_q(T)$  to  $H$ .

The Proposition 3.3 shows that for  $m \in \mathbb{S}$ , the extension is possible only for the operators in slice Hilbert space of  $H$ .

First we show that every  $\mathbb{C}_m$ -Hilbert space is a slice Hilbert space of some quaternionic Hilbert space and for some anti self-adjoint, unitary  $J$ . Later we prove that the extension is also possible for linear operators between two  $\mathbb{C}_m$ -Hilbert spaces.

**Proposition 3.6.** Let  $K$  be a  $\mathbb{C}_m$ -Hilbert space. Then there is a quaternionic Hilbert space  $H$  and an anti self-adjoint, unitary  $J \in \mathcal{B}(H)$  such that

$$K = H_+^{Jm}.$$

*Proof.* Define  $H := K \times K$ . We define binary operations on  $H$  as follows:

$$(x, y) + (z, w) = (x + z, y + w), \text{ for all } (x, y), (z, w) \in H.$$

For  $q \in \mathbb{H}$  we can write  $q = \alpha + \beta \cdot n$ , for some  $\alpha, \beta \in \mathbb{C}_m$ , where  $n \in \mathbb{S}$  is such that  $m \cdot n = -n \cdot m$ . Then

$$(4) \quad (x, y) \cdot (\alpha + \beta \cdot n) = (x \cdot \alpha - y \cdot \beta, x \cdot \beta - y \cdot \alpha).$$

Let us denote the inner product on  $K$  by  $\langle \cdot | \cdot \rangle_K$ . Define  $\langle \cdot | \cdot \rangle : H \times H \rightarrow \mathbb{H}$  by

$$\langle (x, y) | (z, w) \rangle = [\langle x | z \rangle_K + \langle w | y \rangle_K] + [\langle x | w \rangle_K - \langle z | y \rangle_K] \cdot n.$$

Using the fact that  $\langle \cdot | \cdot \rangle_K$  is an inner product on  $K$ , it can be shown that  $\langle \cdot | \cdot \rangle$  is an inner product on  $H$ . The completeness of  $\langle \cdot | \cdot \rangle_K$  implies that  $H$  is complete with respect to the norm,  $\|(x, y)\|^2 = \|x\|^2 + \|y\|^2$  which is induced from the above inner product. Therefore  $H$  is a right quaternionic Hilbert space.

If we identify  $x$  of  $K$  by  $(x, 0)$  of  $H$  and  $y$  of  $K$  by  $(y, 0)$  of  $H$ , then the following identification holds:

$$\begin{aligned} (x, y) &= (x, 0) + (0, y) = (x, 0) + (y, 0) \cdot n \\ &= x + y \cdot n. \end{aligned}$$

Here we used Equation (4) to conclude  $(0, y) = (y, 0) \cdot n$ .

Define  $J: H \rightarrow H$  by

$$J(x + y \cdot n) = (x - y \cdot n) \cdot m, \text{ for all } x + y \cdot n \in H.$$

We shall prove that  $J$  is anti self-adjoint and unitary. If  $x + y \cdot n, z + w \cdot n \in H$  then

$$\begin{aligned} \langle x + y \cdot n | J(z + w \cdot n) \rangle &= \langle x + y \cdot n | (z - w \cdot n) \cdot m \rangle \\ &= \langle x | (z - w \cdot n) \cdot m \rangle + \langle y | (z - w \cdot n) \cdot m \rangle \\ &= \langle x \cdot \overline{m} | z + w \cdot n \rangle + \langle y \cdot \overline{m} \cdot n | z + w \cdot n \rangle \\ &= \langle (-x + y \cdot n) \cdot m | z + w \cdot n \rangle. \end{aligned}$$

Therefore  $J^*(x + y \cdot n) = (-x + y \cdot n) \cdot m$ , for all  $x + y \cdot n \in H$ . This implies  $J^* = -J$  and  $J^*J = JJ^* = I$ .

It remains to show that  $H_+^{Jm} = K$ . If  $x + y \cdot n \in H_+^{Jm}$ , then  $J(x + y \cdot n) = (x + y \cdot n) \cdot m$ . By the definition of  $J$ , it follows that  $y = 0$ . Conversely, if  $x \in K$  then  $J(x) = x \cdot m$ . Hence  $K = H_+^{Jm}$ .  $\square$

From Proposition 3.6 it is clear that for a fixed  $m \in \mathbb{S}$ , a linear operator on any  $\mathbb{C}_m$ - Hilbert space can be extended uniquely to the corresponding quaternionic Hilbert space.

We prove the following lemma to give the relation between the spectrum of  $T$  and the spherical spectrum of  $\tilde{T}$ .

**Lemma 3.7.** *Let  $T: H_+^{Jm} \rightarrow H_+^{Jm}$  be bounded  $\mathbb{C}_m$  - linear and  $\tilde{T}$  be the extension of  $T$  as given in Proposition 3.3. Then*

$$\sigma_S(\tilde{T}) = \{[\lambda] : \lambda \in \sigma(T)\}.$$

*Proof.* If  $\lambda \in \sigma(T)$ , then  $(T - \lambda \cdot I)$  is not invertible. We show that  $\Delta_\lambda(T)$  is not invertible. Suppose  $\Delta_\lambda(T)$  is invertible, there exists  $L \in \mathcal{B}(H)$  such that  $L\Delta_\lambda(T) = \Delta_\lambda(T)L = I$ . This implies  $(T - \overline{\lambda} \cdot I)L = L(T - \overline{\lambda} \cdot I)$ . Equivalently,

$$(T - \lambda \cdot I)[(T - \overline{\lambda} \cdot I)L] = [(T - \overline{\lambda} \cdot I)L](T - \lambda \cdot I) = L.$$

This is contradiction. Therefore  $\Delta_\lambda(T)$  is not invertible. We claim that  $\Delta_\mu(\tilde{T})$  is not invertible for every  $\mu \in [\lambda]$ . If  $\Delta_\mu(\tilde{T})$  is invertible for some  $\mu \in [\lambda]$ , then by Equation (3) and Proposition 3.3(3), we have

$$\Delta_\mu(\tilde{T})^{-1} = \tilde{\Delta}_\mu(T)^{-1} = \tilde{\Delta}_\lambda(T)^{-1}.$$

This is contradiction to  $\lambda \in \sigma(T)$ . Hence

$$\{[\lambda] : \lambda \in \sigma(T)\} \subseteq \sigma_S(\tilde{T}).$$

On the other hand if  $q \in \sigma_s(\tilde{T})$ , then  $\lambda := \text{Re}(q) + m \cdot |\text{Im}(q)| \in [q] \in \sigma_S(\tilde{T})$ . By [7, Proposition 5.11(c)], we have  $\lambda \in \sigma_S(\tilde{T}) \cap \mathbb{C}_m^+ = \sigma(T)$ . Thus

$$\sigma_S(\tilde{T}) = \{[\lambda] : \lambda \in \sigma(T)\}. \quad \square$$

We generalize Proposition 3.3 for linear operators between two different  $\mathbb{C}_m$  - Hilbert spaces.

**Theorem 3.8.** *Let  $H, K$  be quaternionic Hilbert spaces. Let  $J_1, J_2$  be anti self-adjoint unitary operators on  $H$  and  $K$ , respectively. Fix  $m \in \mathbb{S}$ . If  $T: \mathcal{D}(T) \rightarrow K_+^{J_2m}$  is  $\mathbb{C}_m$ - linear with  $\mathcal{D}(T) \subset H_+^{J_1m}$ , then there exists a unique right  $\mathbb{H}$ - linear operator  $\tilde{T}: \mathcal{D}(\tilde{T}) \rightarrow K$  such that*

*$\mathcal{D}(\tilde{T}) \cap H_+^{J_1m} = \mathcal{D}(T)$ ,  $J_1(\mathcal{D}(\tilde{T})) \subset \mathcal{D}(\tilde{T})$  and  $\tilde{T}(u) = T(u)$ , for every  $u \in \mathcal{D}(T)$ . Furthermore,*



- (1) If  $T: H_+^{J_1 m} \rightarrow K_+^{J_2 m}$  is bounded, then  $\tilde{T} \in \mathcal{B}(H, K)$  and  $\|\tilde{T}\| = \|T\|$   
 (2)  $J_2 \tilde{T} = \tilde{T} J_1$  on  $\mathcal{D}(\tilde{T})$ .

On the otherhand, let  $V: \mathcal{D}(V) \rightarrow K$  be right  $\mathbb{H}$ -linear with  $\mathcal{D}(V) \subset H$ . Then  $V = \tilde{U}$ , for  $U: \mathcal{D}(V) \cap H_+^{J_1 m} \rightarrow K_+^{J_2 m}$ , if and only if  $J_2 V = V J_1$  on  $\mathcal{D}(V)$ .

*Proof.* First we show that the extension is unique. Choose  $n \in \mathbb{S}$  such that  $m \cdot n = -n \cdot m$ . Define  $\Phi: H \rightarrow H$  by

$$(5) \quad \Phi(x) = x \cdot n, \text{ for all } x \in H.$$

It is clear that  $\Phi$  is anti  $\mathbb{C}_{m^-}$ -linear isomorphism. Suppose that there exists an extension  $\tilde{T}$  of  $T$  such that  $\mathcal{D}(\tilde{T}) \cap H_+^{J_1 m} = \mathcal{D}(T)$  and  $J_1(\mathcal{D}(\tilde{T})) \subset \mathcal{D}(\tilde{T})$ . Then  $\mathcal{D}(\tilde{T}) \cap H_-^{J_1 m} = \Phi(\mathcal{D}(T))$ . This implies  $\mathcal{D}(\tilde{T}) = \mathcal{D}(T) \oplus \Phi(\mathcal{D}(T))$ . If  $x \in \mathcal{D}(\tilde{T})$ , then  $x = x_1 + x_2$  with  $x_1 \in \mathcal{D}(T)$ ,  $x_2 \in \Phi(\mathcal{D}(T))$  and

$$\tilde{T}(x) = T(x_1) - T(x_2 \cdot n) \cdot n.$$

If  $S: \mathcal{D}(S) \rightarrow K$  is another extension of  $T$  such that  $\mathcal{D}(S) \cap H_+^{J_1 m} = \mathcal{D}(T)$ ,  $J_1(\mathcal{D}(S)) \subset \mathcal{D}(S)$  and  $Sy = Ty$ , for all  $y \in \mathcal{D}(T)$  then  $\mathcal{D}(S) = \mathcal{D}(T) \oplus \Phi(\mathcal{D}(T)) = \mathcal{D}(\tilde{T})$  and we have

$$\begin{aligned} S(x) &= S(x_1) + S(x_2) = S(x_1) - S(x_2 \cdot n) \cdot n \\ &= T(x_1) - T(x_2 \cdot n) \cdot n \\ &= \tilde{T}(x), \text{ for all } x = x_1 + x_2 \in \mathcal{D}(S). \end{aligned}$$

This implies that extension  $\tilde{T}$  of  $T$  is unique.

Proof of (1): If  $T: H_+^{J_1 m} \rightarrow K_+^{J_2 m}$  is bounded then  $\mathcal{D}(\tilde{T}) = H_+^{J_1 m} \oplus H_-^{J_1 m} = H$ . Since  $\tilde{T}$  is the extension of  $T$ , it follows that  $\|T\| \leq \|\tilde{T}\|$ . If  $x = x_1 + x_2 \in H$  then by Lemma 3.2, we have

$$\begin{aligned} \|\tilde{T}x\|^2 &= \|T(x_1) - T(x_2 \cdot n)\|^2 = \|Tx_1\|^2 + \|T(x_2 \cdot n)\|^2 \\ &\leq \|T\|^2(\|x_1\|^2 + \|x_2\|^2) \\ &\leq \|T\|^2\|x\|^2. \end{aligned}$$

This implies that  $\|\tilde{T}\| \leq \|T\|$ . Hence  $\|\tilde{T}\| = \|T\|$ .

Proof of (2): If  $x \in \mathcal{D}(\tilde{T})$ , then  $x = x_1 + x_2$  with  $x_1 \in \mathcal{D}(T)$ ,  $x_2 \in \Phi(\mathcal{D}(T))$  and

$$\begin{aligned} J_2 \tilde{T}(x_1 + x_2) &= J_2[T(x_1) - T(x_2 \cdot n) \cdot n] \\ &= J_2(T(x_1)) - J_2(T(x_2 \cdot n)) \cdot n \\ &= T(x_1) \cdot m - T(x_2 \cdot n) \cdot m \cdot n \\ &= T(x_1 \cdot m) - T(-x_2 \cdot m \cdot n) \cdot n \\ &= T(J_1 x) - T(J_1 x_2 \cdot n) \cdot n \\ &= \tilde{T}(J_1 x_1 + J_1 x_2) \\ &= \tilde{T} J_1(x). \end{aligned}$$

If  $V = \tilde{U}$ , for some  $U: \mathcal{D}(V) \cap H_+^{J_1 m} \rightarrow K_+^{J_2 m}$ , then  $J_1(\mathcal{D}(V)) \subset \mathcal{D}(V)$ . It is clear from the earlier proof that  $\mathcal{D}(V) = \mathcal{D}(U) \oplus \Phi(\mathcal{D}(U))$ . For  $x = x_1 + x_2 \in \mathcal{D}(V)$ , we

have

$$\begin{aligned}
J_2 V(x) &= J_2(Ux_1 - U(x_2 \cdot n) \cdot n) \\
&= U(x_1) \cdot m - U(x_2 \cdot n) \cdot m \cdot n \\
&= U(x_1 \cdot m) + U(x_2 \cdot n \cdot m) \cdot n \\
&= U(J_1 x_1) - U(J_1 x_2 \cdot n) \cdot n \\
&= VJ_1(x).
\end{aligned}$$

Conversely, assume that  $J_2 V = VJ_1$ . That is  $J_2 Vx = VJ_1 x, \forall x \in \mathcal{D}(V)$ . This means that  $J_1 x \in \mathcal{D}(V)$ , for every  $x \in \mathcal{D}(V)$ . Hence  $V(\mathcal{D}(V) \cap H_+^{J_1 m}) \subseteq K_+^{J_2 m}$ .

Define  $U: \mathcal{D}(V) \cap H_+^{J_1 m} \rightarrow K_+^{J_2 m}$  by

$$Ux = Vx, \text{ for all } x \in \mathcal{D}(U).$$

Here  $V$  is right  $\mathbb{H}$ -linear extension of  $U$  such that  $J_1(\mathcal{D}(V)) \subset \mathcal{D}(V)$ , by the uniqueness of extension, we have  $V = \tilde{U}$ .  $\square$

Our aim is to prove that  $L^2(\Omega; \mathbb{C}_m; \mu) = L^2(\Omega; \mathbb{H}; \mu)_+^{J_m}$  for some anti self-adjoint, unitary  $J \in \mathcal{B}(L^2(\Omega; \mathbb{H}; \mu))$ . To establish this result, we need the following theorem.

**Theorem 3.9.** *Let  $m, n \in \mathbb{S}$  be such that  $m \cdot n = -n \cdot m$ . Let  $(\Omega, \mu)$  be a measure space. Then  $L^2(\Omega; \mathbb{C}_m; \mu)$  is closed in  $L^2(\Omega; \mathbb{H}; \mu)$ . More over, we represent  $L^2(\Omega; \mathbb{H}; \mu)$  as  $\{f + g \cdot n | f, g \in L^2(\Omega; \mathbb{C}_m; \mu)\}$ , where*

$$(f + g \cdot n)(x) = f(x) + g(x) \cdot n, \text{ for all } x \in \Omega.$$

*Proof.* If  $f \in L^2(\Omega; \mathbb{H}; \mu)$ , then  $f(x) = \alpha_x + \beta_x \cdot n$ , for every  $x \in \Omega$ , where  $\alpha_x, \beta_x \in \mathbb{C}_m$ . Define  $f_1(x) = \alpha_x$  and  $f_2(x) = \beta_x$ . For  $l = 1, 2$  the map  $f_l: \Omega \rightarrow \mathbb{C}_m$  satisfies

$$\int_{\Omega} |f_l(x)|^2 d\mu(x) \leq \int_{\Omega} |f(x)|^2 d\mu(x) < \infty.$$

Hence  $f_1, f_2 \in L^2(\Omega; \mathbb{C}_m; \mu)$ .

Define  $\Psi: L^2(\Omega; \mathbb{H}; \mu) \rightarrow \{f + g \cdot n | f, g \in L^2(\Omega; \mathbb{C}_m; \mu)\}$  by

$$\Psi(f) = f_1 + f_2 \cdot n, \text{ for every } f \in L^2(\Omega; \mathbb{H}; \mu).$$

Here  $\{f + g \cdot n | f, g \in L^2(\Omega; \mathbb{C}_m; \mu)\}$  is right quaternionic Hilbert space. Clearly,  $\Psi$  is well defined, right  $\mathbb{H}$ -linear onto map. Next, we show that  $\Psi$  is an isometry. Let  $f \in L^2(\Omega; \mathbb{H}; \mu)$ . Then

$$\begin{aligned}
\|\Psi(f)\|^2 &= \int_{\Omega} |f_1(t) + f_2(t) \cdot n|^2 d\mu(t) = \int_{\Omega} |f_1(t)|^2 + |f_2(t)|^2 d\mu(t) \\
&= \int_{\Omega} |f_1(t)|^2 d\mu(t) + \int_{\Omega} |f_2(t)|^2 d\mu(t) \\
&= \|f_1\|^2 + \|f_2\|^2 \\
&= \|f\|^2.
\end{aligned}$$

Thus  $\Psi$  is an isometrically isomorphism. It remains to show that  $L^2(\Omega; \mathbb{C}_m; \mu)$  is closed. Let  $\{f_k\}$  be a sequence in  $L^2(\Omega; \mathbb{C}_m; \mu)$ . If  $\{f_k\}$  converges to  $f = f_1 + f_2 \cdot n$

in  $L^2(\Omega; \mathbb{H}; \mu)$ , then

$$\begin{aligned}\|f_k - f\|^2 &= \|(f_k - f_1) - f_2 \cdot n\|^2 \\ &= \|f_k - f_1\|^2 + \|f_2\|^2.\end{aligned}$$

Since  $f_k \rightarrow f$ , it follows that  $\|f_k - f_1\|^2 \rightarrow 0$ , as  $n \rightarrow \infty$  and  $f_2 = 0$ . Therefore  $L^2(\Omega; \mathbb{C}_m; \mu)$  is closed in  $L^2(\Omega; \mathbb{H}; \mu)$ .  $\square$

**Corollary 3.10.** *There exists  $J \in \mathcal{B}(L^2(\Omega; \mathbb{H}; \mu))$ , an anti self-adjoint, unitary operator such that*

$$L^2(\Omega; \mathbb{H}; \mu)_+^{Jm} = L^2(\Omega; \mathbb{C}_m; \mu).$$

*Proof.* It is clear from Proposition 3.6 and Theorem 3.9 that there exists  $J \in \mathcal{B}(L^2(\Omega; \mathbb{H}; \mu))$  such that

$$L^2(\Omega; \mathbb{H}; \mu)_+^{Jm} = L^2(\Omega; \mathbb{C}_m; \mu). \quad \square$$

#### 4. SPECTRAL THEOREM: BOUNDED OPERATORS

In this section we prove the spectral theorem (multiplication form) for bounded normal operators on a quaternionic Hilbert space.

We recall the spectral theorem for bounded normal operators on complex Hilbert spaces.

**Theorem 4.1.** [4, Theorem 11.5] *If  $N \in \mathcal{B}(H)$  is a normal operator then there is a measure space  $(X, \mu)$  and an essentially bounded  $\mu$ -measurable function  $\phi: X \rightarrow \mathbb{C}$  such that  $N$  is unitarily equivalent to  $L_\phi$ , where  $L_\phi$  is a left multiplication by  $\phi$  acting on  $L^2(X; \mathbb{C}; \mu)$ .*

*More over, if  $H$  is separable then the measure space obtained above  $(X, \mu)$  is  $\sigma$ -finite.*

We prove the multiplication form of normal operator on quaternionic Hilbert spaces by using Proposition 3.3(1), the extension of  $\mathbb{C}_m$ -linear operator on  $H_+^{Jm}$  to the linear operator on quaternionic Hilbert space  $H$ .

#### MULTIPLICATION FORM:.

**Theorem 4.2.** *Let  $T \in \mathcal{B}(H)$  be normal and fix  $m \in \mathbb{S}$ . Then there exists*

- (a) *a Hilbert basis  $\mathcal{N}_m$  of  $H$*
- (b) *a measure space  $(\Omega, \mu)$*
- (c) *a unitary operator  $U: H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  and*
- (d) *an essentially bounded  $\mu$ -measurable function  $\phi: \Omega \rightarrow \mathbb{C}_m$*

*so that if  $T$  is expressed with respect to  $\mathcal{N}_m$ , then*

$$T = U^* M_\phi U,$$

*where  $M_\phi$  is a bounded multiplication operator on  $L^2(\Omega; \mathbb{H}; \mu)$ .*

*Moreover,*

$$(1) \|T\| = \text{ess sup}(|\phi|)$$

$$(2) \sigma_S(T) = \{[\lambda] : \lambda \in \text{ess ran}(\phi)\}.$$

*Further more, if  $H$  is separable Hilbert space, then the obtained measure space  $(\Omega, \mu)$  is  $\sigma$ -finite.*

*Proof.* By Proposition 3.3,  $T_+ : H_+^{Jm} \rightarrow H_+^{Jm}$  is a  $\mathbb{C}_m$ -linear bounded normal operator such that  $\tilde{T}_+ = T$ . Let  $\mathcal{N}_m$  be Hilbert basis for  $H_+^{Jm}$  then by Remark 2.13,  $\mathcal{N}_m$  is Hilbert basis for  $H$  and moreover,

$$J(x) = \sum_{z \in \mathcal{N}} z \cdot m \langle z|x \rangle.$$

If  $u = a + b$ , where  $a \in H_+^{Jm}, b \in H_-^{Jm}$ , then

$$T(u) = T_+(a) - T_+(b \cdot n) \cdot n.$$

By Theorem 4.1, there exist a measure space  $(\Omega, \mu)$ , a  $\mathbb{C}_m$ -valued  $\mu$ -measurable function  $\phi$  on  $\Omega$  and a unitary operator  $U_+ : H_+^{Jm} \rightarrow L^2(\Omega; \mathbb{C}_m; \mu)$  such that

$$T_+ = U_+^* L_\phi U_+,$$

where  $L_\phi : L^2(\Omega; \mathbb{C}_m; \mu) \rightarrow L^2(\Omega; \mathbb{C}_m; \mu)$  is defined by

$$L_\phi(g)(t) = \phi(t) \cdot g(t), \text{ for all } g \in L^2(\Omega; \mathbb{C}_m; \mu).$$

By Theorem 3.3, we have  $\tilde{L}_\phi : L^2(\Omega; \mathbb{H}; \mu) \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  given by

$$\tilde{L}_\phi(g + h \cdot n) = L_\phi(g) + L_\phi(h) \cdot n, \text{ for all } g, h \in L^2(\Omega; \mathbb{C}_m; \mu).$$

Let us denote  $\tilde{L}_\phi$  by  $M_\phi$ . It is clear that  $M_\phi$  is a right  $\mathbb{H}$ -linear and  $M_\phi|_{L^2(\Omega; \mathbb{C}_m; \mu)} = L_\phi$ . For  $h = h_1 + h_2 \cdot n \in L^2(\Omega; \mathbb{H}; \mu)$  and  $x \in \Omega$ , we have

$$\begin{aligned} M_\phi(h_1 + h_2 \cdot n)(x) &= L_\phi(h_1)(x) + L_\phi(h_2)(x) \cdot n \\ &= \phi(x) \cdot h_1(x) + \phi(x) \cdot h_2(x) \cdot n \\ &= \phi(x)(h_1(x) + h_2(x) \cdot n) \\ &= \phi \cdot (h_1 + h_2)(x). \end{aligned}$$

That is  $M_\phi$  is a multiplication operator induced by  $\phi$ . By Theorem 3.8,  $U_+$  has a unique extension  $U : H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  such that

$$U(a + b) = U_+(a) - U_+(b \cdot n) \cdot n, \text{ for all } a \in H_+^{Jm}, b \in H_-^{Jm}.$$

Therefore, for all  $a \in H_+^{Jm}$  and  $b \in H_-^{Jm}$ , we have

$$\begin{aligned} U^* M_\phi U(a + b) &= U^* M_\phi(U_+(a) - U_+(b \cdot n) \cdot n) \\ &= U^* [L_\phi(U_+(a)) - L_\phi(U_+(b \cdot n)) \cdot n] \\ &= U_+^* L_\phi U_+(a) + U_+^* (L_\phi U_+(b \cdot n) \cdot n \cdot n) \cdot n \\ &= U_+^* L_\phi U_+(a) - U_+^* L_\phi U_+(b \cdot n) \cdot n \\ &= T_+(a) - T_+(b \cdot n) \cdot n \\ &= T(a + b). \end{aligned}$$

Hence  $U^* M_\phi U = T$ .

Proof of (1): It is clear from Proposition 3.3(1), that  $\|T\| = \|T_+\|$  and since  $\|T_+\| = \text{ess sup}(|\phi|)$ , we have  $\|T\| = \text{ess sup}(|\phi|)$ .

Proof of (2): We know that  $\sigma(T_+) = \text{ess ran}(\phi)$ . By Lemma 3.7, we have

$$\sigma_S(T) = \{[\lambda] : \lambda \in \text{ess ran}(\phi)\}$$

If  $H$  is separable, by Lemma 3.1  $H_+^{Jm}$  is separable. Therefore by Theorem 4.1,  $(\Omega, \mu)$  is  $\sigma$ -finite.  $\square$

**Corollary 4.3.** *Let  $T \in \mathcal{B}(H)$  be anti self-adjoint operator and fix  $m \in \mathbb{S}$ . Then, there exists Hilbert basis  $\mathcal{N}_m$ , a measure space  $(\Omega; \mu)$ , a  $\mu$  - measurable purely imaginary function  $\xi: \Omega \rightarrow \mathbb{C}_m$  and a unitary operator  $U: H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  so that if  $T$  is expressed with respect to  $\mathcal{N}_m$  then*

$$T = U^* M_\xi U.$$

*In particular,  $T$  is unitary  $\Leftrightarrow |\xi| = 1$  in  $\mu$ - a.e.*

*Proof.* By Theorem 4.2, we have  $T = U^* M_\xi U$ . Since  $T$  is anti self-adjoint,  $T^* = -T$ . It follows that  $U^* M_{\bar{\xi}} U = U^* M_{-\xi} U$ .

This implies  $\bar{\xi} = -\xi$ . In particular,

$$T \text{ is unitary} \Leftrightarrow T^* T = T T^* = I$$

$$\Leftrightarrow U^* M_{|\xi|^2} U = I$$

$$\Leftrightarrow U^* M_{|\xi|^2} U = U^* M_1 U \text{ (Here } M_1 \text{ is the identity on } L^2(\Omega; \mathbb{H}; \mu))$$

$$\Leftrightarrow |\xi| = 1, \quad \mu \text{ - a.e.} \quad \square$$

**Corollary 4.4.** *Let  $T \in \mathcal{B}(H)$  be normal. Then  $T$  and  $T^*$  are unitarily equivalent.*

*Proof.* By Theorem 4.2, there exists a measure space  $(\Omega, \mu)$ ,  $\mu$ - measurable  $\mathbb{C}_m$ -valued function  $\xi$  on  $\Omega$  and a unitary  $U: H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  such that  $T = U^* M_\xi U$ . This implies  $T^* = U^* M_{\bar{\xi}} U$ .

For  $x \in \Omega$ , define  $\phi(x) = \frac{\xi(x) \cdot n}{|\xi(x)|}$ , for all  $x \in \Omega$ . Here  $n \in \mathbb{S}$  such that  $m \cdot n = -n \cdot m$ . Clearly,  $\phi$  is non-zero a.e ( $\mu$ ),  $\text{ess sup } (|\phi|) = 1$  and  $M_\phi$  is a right linear, unitary operator. Moreover,  $M_\phi^* M_{\bar{\xi}} M_\phi = M_{\bar{\xi}}$ . Hence,

$$\begin{aligned} T &= U^* M_\xi U = U M_\phi^* M_{\bar{\xi}} M_\phi U \\ &= U M_\phi^* U T^* U^* M_\phi U \end{aligned}$$

Let  $V = U M_\phi^* U$ . Then  $V$  is unitary and  $T = V T^* V^*$ . Hence  $T$  and  $T^*$  are unitarily equivalent.  $\square$

**Example 4.5.** *Let  $\phi(t) = (i - j - k)t$ , for all  $t \in [0, 1]$ . Then  $\phi$  is essentially bounded measurable function with the Lebesgue measure  $\mu$  on  $[0, 1]$ . Define  $M_\phi: L^2([0, 1]; \mathbb{H}; \mu) \rightarrow L^2([0, 1]; \mathbb{H}; \mu)$  by*

$$M_\phi(g)(t) = \phi(t) \cdot g(t) \text{ for all } g \in L^2([0, 1]; \mathbb{H}; \mu).$$

*We show that  $M_\phi$  is unitarily equivalent to a multiplication operator on  $L^2([0, 1]; \mathbb{H}; \mu)$  induced by some complex valued measurable function.*

*Define  $U: L^2([0, 1]; \mathbb{H}; \mu) \rightarrow L^2([0, 1]; \mathbb{H}; \mu)$  by*

$$U(g)(t) = \frac{(\sqrt{3} + 1) - j + k}{\sqrt{6 + 2\sqrt{3}}} \cdot g(t), \text{ for all } g \in L^2([0, 1]; \mathbb{H}; \mu).$$

*It follows that*

$$U^*(h)(t) = \frac{(\sqrt{3} + 1) + j - k}{\sqrt{6 + 2\sqrt{3}}}, \text{ for all } h \in L^2([0, 1]; \mathbb{H}; \mu).$$

*It can be easily verified that  $U$  is unitary.*

*Define  $\eta(t) = \sqrt{3}it$  for all  $t \in [0, 1]$ . Clearly,  $\eta$  is a complex valued essentially bounded measurable function. Also  $\eta$  induces a bounded multiplication operator*

$M_\eta$  on  $L^2([0, 1]; \mathbb{H}; \mu)$ . We prove that  $M_\phi$  is unitarily equivalent to  $M_\eta$ . For all  $g \in L^2([0, 1]; \mathbb{H}; \mu)$ , we have

$$\begin{aligned}
U^* M_\eta U(g)(t) &= U^* M_\eta \frac{(\sqrt{3}+1) - j + k}{\sqrt{6+2\sqrt{3}}} \cdot g(t) \\
&= U^* \sqrt{3}it \cdot \frac{(\sqrt{3}+1) - j + k}{\sqrt{6+2\sqrt{3}}} \cdot g(t) \\
&= \frac{(\sqrt{3}+1) + j - k}{\sqrt{6+2\sqrt{3}}} \cdot \sqrt{3}it \cdot \frac{(\sqrt{3}+1) - j + k}{\sqrt{6+2\sqrt{3}}} \cdot g(t) \\
&= (i - j - k)t \cdot g(t) \\
&= M_\phi(g)(t).
\end{aligned}$$

This example 4.5 infers that every multiplication operator induced by quaternion valued function is unitarily equivalent to a multiplication operator induced by some complex valued function.

## 5. SPECTRAL THEOREM: UNBOUNDED OPERATORS

In this section we prove the spectral theorem ( multiplication form ) for unbounded normal operator in a quaternionic Hilbert space. For this, we restrict the operator to the slice Hilbert space, associate a bounded operator to the restricted operator through the  $\mathcal{Z}$ - transform then by using Theorem 4.1, we obtain the result. Analogous to the complex Hilbert spaces, the theory of unbounded operators in quaternionic Hilbert space is studied in [7]. We recall some properties of unbounded operators in quaternionic Hilbert space, that we need for our purpose.

**Definition 5.1.** [10, Definition 13.1] *Let  $T: \mathcal{D}(T) \subset H_1 \rightarrow H_2$  be a right linear operator with domain  $\mathcal{D}(T)$ , a right linear subspace of  $H_1$ . Then  $T$  is said to be densely defined, if  $\mathcal{D}(T)$  is dense in  $H_1$ .*

The graph of an operator  $T$  is denoted by  $\mathcal{G}(T)$  and is defined as

$$\mathcal{G}(T) = \{(x, Tx) | x \in \mathcal{D}(T)\}.$$

**Definition 5.2.** *Let  $T: H_1 \rightarrow H_2$  be right linear with domain  $\mathcal{D}(T) \subset H_1$ . Then  $T$  is said to be closed, if the graph  $\mathcal{G}(T)$  is closed in  $H_1 \times H_2$ . Equivalently, if  $(x_n) \subset \mathcal{D}(T)$  with  $x_n \rightarrow x \in H_1$  and  $Tx_n \rightarrow y$ , then  $x \in \mathcal{D}(T)$  and  $Tx = y$ .*

Let  $S, T$  be densely defined closed operators with domains  $\mathcal{D}(S), \mathcal{D}(T)$ , respectively. Then  $S$  is said to be a restriction of  $T$  denoted by  $S \subset T$ , if  $\mathcal{D}(S) \subseteq \mathcal{D}(T)$  and  $Sx = Tx$ , for all  $x \in \mathcal{D}(S)$ . In this case,  $T$  is called an extension of  $S$ .

**Definition 5.3.** *An operator  $T$  with domain  $\mathcal{D}(T)$  is said to be normal, if  $T$  is densely defined, closed and  $T^*T = TT^*$ .*

**Definition 5.4.** *Let  $T: \mathcal{D}(T) \rightarrow H$  be densely defined closed operator and  $S \in \mathcal{B}(H)$ , then we say that  $S$  commute with  $T$  if  $ST \subseteq TS$ . In other words,  $Sx \in \mathcal{D}(T)$  and  $STx = TSx$ , for all  $x \in \mathcal{D}(T)$ .*

Recall that if  $T \in \mathcal{B}(H)$  is normal, then by Remark 3.4 there exists a  $J \in \mathcal{B}(H)$  anti self-adjoint, unitary such that  $TJ = JT$ . We generalize this result to the case of densely defined, closed right linear operators.

**Theorem 5.5.** [2, Theorem 6.1] *Let  $T: \mathcal{D}(T) \rightarrow H$  be densely defined closed, right linear operator. Define the operator  $\mathcal{Z}_T := T(I + T^*T)^{-\frac{1}{2}}$ . Then  $\mathcal{Z}_T$  has the following properties:*

- (1)  $\mathcal{Z}_T \in \mathcal{B}(H)$ ,  $\|\mathcal{Z}_T\| \leq 1$  and  $T = \mathcal{Z}_T(I - \mathcal{Z}_T^*\mathcal{Z}_T)^{-\frac{1}{2}}$
- (2)  $(\mathcal{Z}_T)^* = \mathcal{Z}_T^*$
- (3) If  $T$  is normal, then  $\mathcal{Z}_T$  is normal.

**Theorem 5.6.** *Let  $T: \mathcal{D}(T) \rightarrow H$  be normal with  $\mathcal{D}(T) \subset H$ . Then there exists an anti self-adjoint unitary operator  $J \in \mathcal{B}(H)$  such that  $J$  commutes with  $T$ .*

*Proof.* It is clear from the Theorem 5.5, that  $\mathcal{Z}_T$  is a bounded right linear normal operator. By Proposition 2.11, there exists an anti self-adjoint and unitary operator  $J \in \mathcal{B}(H)$  such that  $\mathcal{Z}_T J = J \mathcal{Z}_T$  and  $J \mathcal{Z}_T^* = \mathcal{Z}_T^* J$ . This implies that  $J(I - \mathcal{Z}_T^* \mathcal{Z}_T) = (I - \mathcal{Z}_T^* \mathcal{Z}_T) J$ . So  $J$  commutes with the square roof of bounded positive operator  $(I - \mathcal{Z}_T^* \mathcal{Z}_T)$ , that is  $J(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}} = (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}} J$ .

Now we show that  $J$  commutes with the inverse of  $(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}}$ , which is an unbounded operator. Let  $x \in \mathcal{D}((I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}})$ . Then  $x = (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}} y$ , for some  $y \in \mathcal{D}((I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}})$  and  $Jx = J(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}} y = (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{\frac{1}{2}} Jy$ . This implies  $Jx \in \mathcal{D}((I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}})$ . Moreover,

$$J(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}} x = Jy = (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}} Jx.$$

It is enough to show that  $JT \subseteq TJ$ . Since  $T = \mathcal{Z}_T(I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}}$  and  $\mathcal{D}(T) = \mathcal{D}((I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}})$ , for every  $x \in \mathcal{D}(T)$ , we have  $Jx \in \mathcal{D}(T)$  and

$$\begin{aligned} JT x &= J \mathcal{Z}_T (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}} x = \mathcal{Z}_T J (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}} x \\ &= \mathcal{Z}_T (I - \mathcal{Z}_T^* \mathcal{Z}_T)^{-\frac{1}{2}} Jx \\ &= TJ x. \end{aligned}$$

Hence the result.  $\square$

**Lemma 5.7.** *Let  $H$  be a quaternionic Hilbert space and  $J$  is an anti self-adjoint and unitary. If  $T: \mathcal{D}(T) \subset H_+^{Jm} \rightarrow H_+^{Jm}$  is  $\mathbb{C}_m$ -linear, then  $\tilde{\mathcal{Z}}_T = \mathcal{Z}_{\tilde{T}}$ .*

*Proof.* By Theorem 5.6, there exists an anti self-adjoint, unitary operator  $J \in \mathcal{B}(H)$  such that  $JT \subseteq TJ$ . By Proposition 3.3,

$$\begin{aligned} \tilde{\mathcal{Z}}_T &= \tilde{T}(\tilde{I}_{H_+^{Jm}} + \tilde{T}^* \tilde{T})^{-\frac{1}{2}} \\ &= \tilde{T}(I_H + \tilde{T}^* \tilde{T})^{-\frac{1}{2}} \\ &= \mathcal{Z}_{\tilde{T}}. \end{aligned}$$

It is enough to show that  $\widetilde{(I_{H_+^{Jm}} + T^* T)} = (I_H + \tilde{T}^* \tilde{T})$ . We have

$$\begin{aligned} \widetilde{(I_{H_+^{Jm}} + T^* T)}(x + y) &= (I_{H_+^{Jm}} + T^* T)(x) - (I_{H_+^{Jm}} + T^* T)(y \cdot n) \cdot n \\ &= (x + y) + \widetilde{T^* T}(x + y) \\ &= (I_H + \tilde{T}^* \tilde{T})(x + y), \end{aligned}$$

for all  $x + y \in \mathcal{D}(\widetilde{(I_{H_+^{Jm}} + T^* T)})$ .  $\square$

**Theorem 5.8.** *Let  $T: \mathcal{D}(T) \rightarrow H$  be normal and fix  $m \in \mathbb{S}$ . Then there exists*

- (a) a Hilbert basis  $\mathcal{N}_m$  of  $H$
- (b) a measure space  $(\Omega, \mu)$
- (c) a unitary operator  $U: H \rightarrow L^2(\Omega; \mathbb{H}; \mu)$  and
- (d) a  $\mu$ -measurable function  $\phi: \Omega \rightarrow \mathbb{C}_m$

so that if  $T$  is expressed with respect to  $\mathcal{N}_m$  then

$$Tx = U^* M_\eta Ux, \text{ for all } x \in \mathcal{D}(T),$$

where  $M_\eta$  is right  $\mathbb{H}$ -linear multiplication operator acting on  $L^2(\Omega_0; \mathbb{H}; \nu)$  with domain

$$\mathcal{D}(M_\eta) = \{g \in L^2(\Omega_0; \mathbb{H}; \nu) \mid \eta \cdot g \in L^2(\Omega_0; \mathbb{H}; \nu)\}.$$

More over, if  $H$  is separable then the measure space obtained  $(\Omega_0; \nu)$  is  $\sigma$ -finite.

*Proof.* By Theorem 3.3, there exists a unique  $T_+: \mathcal{D}(T) \cap H_+^{Jm} \rightarrow H_+^{Jm}$ , a  $\mathbb{C}_m$ -linear operator such that  $\tilde{T}_+ = T$ . Let  $\mathcal{N}_m$  be Hilbert basis for  $H_+^{Jm}$  then by Remark 2.13,  $\mathcal{N}_m$  is Hilbert basis for  $H$  and moreover,

$$J(x) = \sum_{z \in \mathcal{N}} z \cdot m \langle z | x \rangle.$$

It is clear that  $\tilde{\mathcal{Z}}_{T_+} = \mathcal{Z}_T$  and  $\mathcal{Z}_{T_+}$  is bounded  $\mathbb{C}_m$ -linear operator. By Theorem 4.1, there is a measure space  $(\Omega; \mu)$ , a unitary operator  $U_+: H_+^{Jm} \rightarrow L^2(\Omega; \mathbb{C}_m; \mu)$  and a  $\mu$ -measurable function  $\phi$  such that

$$(6) \quad \mathcal{Z}_{T_+} = U_+^* L_\phi U_+.$$

Here  $\Omega = \sigma(\mathcal{Z}_{T_+})$ . Define  $\xi: \Omega \rightarrow \mathbb{C}_m$  by

$$\xi(p) = p(1 - |p|^2)^{-\frac{1}{2}}, \text{ for all } p \in \Omega.$$

Then  $\xi$  is  $\mu$ -measurable function such that

$$\mu(\{x \in \Omega : \xi(x) = \infty\}) = 0$$

By the Borel functional calculus for bounded  $\mathbb{C}_m$ -linear operator  $\mathcal{Z}_{T_+}$ , we write

$$(7) \quad \xi(\mathcal{Z}_{T_+}) = \mathcal{Z}_{T_+}(I - \mathcal{Z}_{T_+}^* \mathcal{Z}_{T_+})^{-\frac{1}{2}} = T_+.$$

By Equations (6) and (7), we have

$$\begin{aligned} T_+ &= U_+^* L_\phi U_+ (I - U_+^* L_{|\phi|^2} U_+)^{-\frac{1}{2}} \\ &= U_+^* L_{\phi(1-|\phi|^2)^{-\frac{1}{2}}} U_+. \end{aligned}$$

Let us denote  $\psi = \phi(1 - |\phi|^2)^{-\frac{1}{2}}$ . Then

$$(8) \quad T_+ x = U_+^* L_\psi U_+ x, \text{ for all } x \in \mathcal{D}(T_+).$$

This implies that  $U_+(\mathcal{D}(T_+)) \subset \mathcal{D}(L_\psi)$ .

It is clear  $\sigma(T_+) = \text{ess ran}(\psi) = \Omega_0$  (say). Define a measure on  $\Omega_0$  as  $\nu(S) = \mu(\xi^{-1}(S))$ , for every Borel subset  $S$  in  $\Omega_0$ .

If  $\eta(z) = z$  on  $\Omega_0$ , then  $L_\eta: \mathcal{D}(L_\eta) \rightarrow L^2(\Omega_0; \mathbb{C}_m; \nu)$  defines a right linear operator in  $L^2(\Omega; \mathbb{C}_m; \mu)$  with the domain  $\mathcal{D}(L_\eta) = \{g \in L^2(\Omega_0; \mathbb{C}_m; \nu) \mid \eta \cdot g \in L^2(\Omega_0; \mathbb{C}_m; \nu)\}$ . We establish a unitary between  $L^2(\Omega; \mathbb{C}_m; \mu)$  and  $L^2(\Omega_0; \mathbb{C}_m; \nu)$  as follows:

Define  $\pi: L^2(\Omega; \mathbb{C}_m; \mu) \rightarrow L^2(\Omega_0; \mathbb{C}_m; \nu)$  by

$$\pi(g) = g \circ \xi^{-1}, \text{ for all } g \in L^2(\Omega; \mathbb{C}_m; \mu).$$



We claim that  $\pi$  is a unitary. For  $g \in L^2(\Omega; \mathbb{C}_m; \mu)$ , we have

$$\begin{aligned} \|\pi(g)\|^2 &= \int_{\Omega_0} |(g \circ \xi^{-1})(s)|^2 d\nu(s) \\ &= \int_{\Omega_0} |g(\xi^{-1}(s))|^2 d\nu(s) \\ &= \int_{\Omega} |g(t)|^2 d\mu(t) \\ &= \|g\|^2. \end{aligned}$$

This shows that  $\pi$  is one to one. If  $h \in L^2(\Omega_0; \mathbb{C}_m; \nu)$ , then  $h \circ \xi \in L^2(\Omega; \mathbb{C}_m; \mu)$  and  $\pi(h \circ \xi) = h$ .

Let  $g \in L^2(\Omega; \mathbb{C}_m; \mu)$  and  $h \in L^2(\Omega_0; \mathbb{C}_m; \nu)$ . Then

$$\begin{aligned} \langle \pi(g) | h \rangle &= \int_{\Omega_0} \overline{\pi(g)(x)} h(x) d\nu(x) \\ &= \int_{\Omega_0} \overline{g(\xi^{-1}x)} h(x) d\nu(x) \\ &= \int_{\Omega} g(y) h(\xi(y)) d\mu(y) \\ &= \langle g | h \circ \xi \rangle. \end{aligned}$$

This implies  $\pi^*(h) = h \circ \xi$ , for all  $h \in L^2(\Omega_0; \mathbb{C}_m; \nu)$ . It can be verified that  $\pi^*\pi = \pi\pi^* = I$ . First we express  $L_\eta$  in terms of  $L_\psi$ . Later we construct unitary  $V_+$  between  $H_+^{Jm}$  and  $L^2(\Omega_0; \mathbb{C}_m; \nu)$ . Consider

$$\begin{aligned} (\pi^* L_\eta \pi)(g)(x) &= \pi^* L_\eta (g \circ \xi^{-1})(x) \\ &= \pi(\eta \cdot g \circ \xi^{-1})(x) \\ &= \eta \cdot (g \circ \xi^{-1}) \circ \xi(x) \\ &= (\eta \circ \xi)(x) \cdot g(x) \\ &= \psi(x) \cdot g(x) \\ &= L_\psi(g)(x). \end{aligned}$$

This shows  $\pi^* L_\eta \pi = L_\psi$ .

Define  $V_+ : H_+^{Jm} \rightarrow L^2(\Omega_0; \mathbb{C}_m; \nu)$  by

$$V_+ = \pi \circ U_+.$$

The following diagram helps in understanding the construction of unitary operators.

$$\begin{array}{ccccc}
 & & \mathcal{D}(L_\psi) \subseteq L^2(\Omega; \mathbb{C}_m; \mu) & \xrightarrow{L_\psi} & L^2(\Omega; \mathbb{C}_m; \mu) \\
 & \nearrow U_+ & \downarrow \pi & & \uparrow \pi^* \\
 H_+^{Jm} & \xrightarrow{V_+} & \mathcal{D}(L_\eta) \subseteq L^2(\Omega_0; \mathbb{C}_m; \nu) & \xrightarrow{L_\eta} & L^2(\Omega_0; \mathbb{C}_m; \nu)
 \end{array}$$

We claim that  $V_+$  is unitary and  $V_+^* L_\eta V_+ u = T_+ u$  for all  $u \in \mathcal{D}(T_+)$ . It can be easily seen that  $V_+^* V_+ = V_+ V_+^*$ . Then by Equation (8), we have for all  $u \in \mathcal{D}(T_+)$ ,

$$\begin{aligned}
 T_+ u &= U_+^* \pi^* L_\eta \pi U_+ u \\
 &= V_+^* L_\eta V_+ u.
 \end{aligned}$$

Now extend the operator  $T_+$  to the operator  $T$  in  $H$  by using Theorem 3.3 and Theorem 3.8. The rest of the proof follows in the similar lines as in the case of bounded operators. Let  $\widetilde{L}_\eta = M_\eta$  and  $\widetilde{V}_+ = V$ . Then by extension of  $T_+$  we get

$$Tx = VM_\eta Vx, \text{ for all } x \in \mathcal{D}(T).$$

where  $M_\eta$  is the multiplication operator in  $L^2(\Omega_0; \mathbb{H}; \nu)$ .

If  $H$  is separable, then  $\mu$  is  $\sigma$ -finite by Theorem 4.2. Therefore  $\nu$  is also  $\sigma$ -finite.  $\square$

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